STOCHASTIC INVESTMENT HORIZONS IN THE ASSET ALLOCATION DECISION AND LIABILITY DRIVEN INVESTING

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June 27, 2014

1) Target returns — The motivation

According to the theory of quantum mechanics, in nature there exist sets of "complementary variables", such as the position and momentum of a particle, such that any attempt to precisely measure one of the variables will result to complete loss of information on the other.

In the financial markets the corresponding set of complementary variables is the time period of holding a risky asset and its holding period rate of return. If one fixes the holding period of investing in an asset, (a fixed horizon "FH" investor), then the holding period rate of return is a random variable. If on the other hand, one fixes the rate of return to be produced by investing in an asset, (a target return "TR" investor), then the holding period to achieve such return is uncertain.

In what follows we will examine the implications of these two complementary approaches to the structure of the risk premium offered by risky assets, the asset allocation decision between risky and risk free assets, and explore in more detail a target return example from the area of liability driven investments.

2) Fixed horizon — Stochastic rate of return

We assume that the instantaneous return of a risky asset follows a random walk with drift μ (equal to the constant risk free rate r, plus a risk premium δ), and a constant volatility σ. All rates of return are annualized with continuous compounding. We let Y stand for the dollar price of the asset, and assume no taxes, trading costs, or dividend yields to keep the discussion simple.

The dynamics of Y is described by:

$$dY = (r + \delta)Ydt + \sigma YdZ \qquad (1)$$

where Z stands for the standard Brownian motion.

The utility u of the asset excess return is:

$$u = \langle EROR \rangle - \frac{1}{2} aVar(EROR) \qquad (2)$$

The brackets stand for the expectation operator and *"Var"* for the variance of the variable in the parenthesis. a is the risk aversion of the investor and *EROR* is the excess rate of return of the risky over that of the risk free asset.

The percentage of the FH investor's wealth γ, to be allocated to the risky asset is:

$$\gamma = \frac{\langle EROR \rangle}{aVar(EROR)} \qquad (3)$$

With the societal risk aversion A the weighted average of the risk aversions of individual investors, then the utility U that the society places on the excess returns of the asset is:

$$U = \langle EROR \rangle - \frac{1}{2} AVar(EROR) = 0 \qquad (4)$$

in order for markets to clear.

In a market where risk premia are set by FH investors, the risk premium of the market is obtained by substituting in (4):

$$\varepsilon = \frac{1}{2} A\sigma^2 \qquad (5)$$

where, $\varepsilon = \delta - \sigma^2/2$. Substituting in (3), we obtain that in equilibrium, a FH investor will allocate:

$$\gamma_{FH} = \frac{\varepsilon}{\alpha\sigma^2} = \frac{1}{2}\frac{A}{\alpha} \qquad (6)$$

percent of the portfolio to the risky asset.

3) Fixed rate of return — Stochastic horizon

A TR investor will make asset allocation decisions based on establishing a target rate of return R that he wishes to achieve by investing in the risky asset. This introduces an investment horizon uncertainty. The investor knows for certainty that the return from the risky asset will be R, but it is not known in advance how much time t, it will take for the return to hit R. This investor will make investment decisions based on the distribution of t, instead of the distribution of returns that a FH investor employs. Specifically, the expected length of the holding period and its variance will be of importance to his investment decisions.

If the asset price follows (1), then:

$$< t > = \frac{2R}{\sigma^2(\rho-1)} = \frac{R}{\mu-\sigma^2/2} \qquad (7)$$

$$Var(t) = \frac{8R}{(\rho-1)^3\,\sigma^4} = \frac{R\sigma^2}{(\mu-\frac{\sigma^2}{2})^3} \qquad (8)$$

where: $\rho=2\mu/\sigma^2>1$.

In this framework, the excess return of investing in the risky asset over the risk free return is given by

$$EROR = R - rt \qquad (9)$$

The uncertainty of the excess return has been transferred to the term with the risk free rate. What is variable here is not the risky asset *ROR*, as this has been fixed at *R*, but the risk-free *ROR* over the variable holding period. Taking expectations and variances in (9), and substituting in (4), we obtain the following equation for the risk premium:

$$\varepsilon(r+\varepsilon)^2 - \frac{1}{2}\,Ar^2\sigma^2 = 0 \qquad (10)$$

4) Properties of the TR risk premium

From (10) we see that the risk premium ε is no longer a linear function of *A* and σ^2, which was the case in (5).

The power series expansion of ε produces:

$$\varepsilon = \frac{1}{2}\sigma^2 A - \frac{1}{2}\frac{\sigma^4}{r}A^2 + \frac{7\sigma^6}{8r^2}A^3 + O(\sigma^2 A)^{7/2} \qquad (11)$$

where $O(...)^k$ means that the terms not shown are of the order *k* for the variable in the parenthesis. The first term is the risk premium for a FH investor. When an asset is not very volatile, the investor is not risk averse, or risk free rates are very high, the two approaches of assigning risk premia are equivalent.

We see that the TR risk premium increases with *A*, *r* and σ. The dependence on *r* is a new feature of the TR risk premium. When the risk-free rate is high, it would take a higher risk premium, all else being the same, to entice the TR investor into the risky asset, since a high risk-free rate increases the amount of the shortfall should the hitting time for *R* by the risky asset turns out to be greater than *T*.

The TR investor will always demand a lower risk premium than the FH investor. This can provide an explanation to the fact that over long time periods the currency markets appear to offer low risk premia, relative to what would have been expected based on their volatilities. If one assumes that currency markets are dominated by multinational corporations that desire to fix their currency exposures by engaging in futures or currency swap transactions, then the currency market risk premia are set by TR investors, and as such are expected to be lower than premia set in FH investor dominated markets.

During time periods, such as the current, when risk premia collapse without a commensurate decrease in volatilities, the decline is usually attributed to a lower societal risk aversion *A*. We also suggest that a decline in risk premia can occur with unchanged *A*, but with an increasing relative percent of investors employing TR strategies, that will demand lower risk premia.

After a straightforward calculation, we obtain that for the TR investor, the percent allocated to the risky asset, when *market risk premia are set by FH* investors is:

$$\gamma_{TR} = \frac{1}{2}\frac{A}{\alpha}\left(1+\frac{1}{2}\,\frac{A\sigma^2}{r}\right)^2 \xrightarrow{\alpha=A} \frac{1}{2}\left(1+\frac{A\sigma^2}{2r}\right)^2 \qquad (12)$$

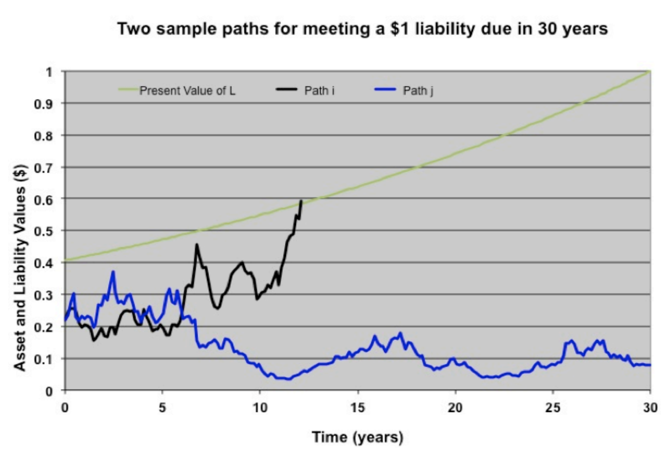
An investor following a TR strategy, in a market where risk premia are set by FH investors will always allocate more to the risky asset than the FH counterpart with the same risk aversion. This is expected, since the TR investor requires lower risk premia than his FH counterpart, and the excess risk premium offered in the FH world causes him to allocate more to the risky asset. This is especially true in a low risk-free rate world. The target return investor now faces a lower opportunity cost, as the risk free rate is low and can allocated more to the risky asset.

5) Stochastic horizons in Liability Driven Investing

Liability driven investing is another area with stochastic investment horizon applications. We will analyze the simplest case, that of a liability *L*, due in time *T*, with a constant discounting factor of *r*, across all maturities. More general problems can be analyzed, but numerical techniques must then be employed.

When currently one has an asset worth *$A*, that is less than the present value of *L*, the plan is underfunded. In order to meet *L*, one would have to invest in a risky asset with drift $r+\delta$ and volatility σ. If at some future time *t*, the value of *A* meets the present value of *L* at that time, *(PVL,t)*, one can then dedicate the portfolio and achieve fully funded status.

Below is an example of a liability *L* of $1 due in 30 years, with *r*=3%, and two sample paths of a risky asset with drift of 6% and a volatility of 30%. *L* has a present value of almost 40 cents today, which will increase over time exponentially, as the discounting period decreases. Sample path *i* crosses the *PVL* curve a little after 10 years, at which point the portfolio is dedicated and the liability is fully funded. Sample path *j* never crosses the *PVL* curve and *L* remains underfunded.



The present value of the liability L due in T is given by:

$$PVL(t) = L \exp(-r(T-t)) \qquad (13)$$

To present an analytic solution, we assume that the risky asset follows the scaled Brownian motion with drift:

$$X_t = X_o + \mu t + \sigma W_t \qquad (14)$$

with W_t the standard Brownian motion, X_t the log-price of the asset, and

$$X_o = -\ln(Le^{-rT} / A) \qquad (15)$$

the cumulative, continuously compounded rate of return needed on top of the return of the risk-free rate to cover the shortfall. In this framework, the exponential curve for the present value of L ("*PVL* curve") becomes a straight line with slope r, starting at the origin, and the problem at hand is to find the probability distribution function of the first passage time τ for X_t=0, with X_0 given in (15), and $\mu=\delta$.

This is a standard problem in the theory of stochastic processes, with the probability of the first hitting time of the *PVL* curve being between t and $t+dt$ given by:

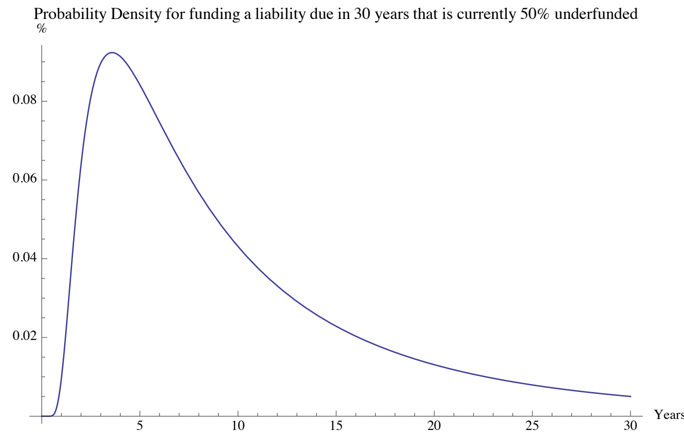
$$P(\tau \in (t, t+dt)) = \frac{|X_0|}{\sigma\sqrt{2\pi t^3}} e^{-(|X_0|-\delta t)^2/2\sigma^2 t} dt \qquad (16)$$

From (16) we can obtain the expected passage time: $|X_0|/\delta$, and the variance of the first passage time: $|X_0|\sigma^2/\delta^3$. The maximum of the density function is:

$$\tau_{max} = \frac{X_0^2}{3\sigma^2} \qquad (17)$$

If one uses a high volatility asset to cover the underfunding, then the investor will know quite soon whether he is successful, as the most likely time is inversely proportional to the volatility, and is independent of the drift of the process. Also, the most likely time depends on the return needed per risk added, i.e. on the ratio of X_0 to σ.

Below is the graph for the probability density for funding a liability due in 30 years that is currently 50% underfunded, by investing in an asset with risk premium of 6% and volatility of 20%.



Conclusion: Use of hitting time probabilities for risk management

Numerical methods must be applied to address the more complex problems of: a general term structure of interest rates, of stochastic volatilities for both the discounting factors as well as the risky assets and their correlations, and the fact that the log-normally distributed returns we assumed thus far, are but an idealization, and returns often exhibit jumps, fatter tails and are asymmetric. Even with the simplified assumptions we used, necessary to arrive at analytical solutions, the use of hitting time probabilities offers an indispensable tool to ask relevant questions, obtain meaningful answers and intelligently manage the risks involved in asset-liability investing.

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